Nonperturbative gravito-magnetic fields

Carlos F. Sopuerta†§, Roy Maartens‡, George F.R. Ellis*, and William M. Lesame††

† *Department of Fundamental Physics, University of Barcelona, E-08028 Barcelona, Spain*
§ *Institute for Theoretical Physics, FSU Jena, Max-Wien-Platz 1, D-07743 Jena, Germany*
‡ *School of Computer Science and Mathematics, Portsmouth University, Portsmouth PO1 2EG, England*
Department of Mathematics and Applied Mathematics, University of Cape Town, Cape Town 7701, South Africa
†† *Department of Mathematics, Applied Mathematics and Astronomy, University of South Africa, Pretoria 0001, South Africa*

(15 April 1999)

In a cold matter universe, the linearized gravito-magnetic tensor field satisfies a transverse condition (vanishing divergence) when it is purely radiative. We show that in the nonlinear theory, it is no longer possible to maintain the transverse condition, since it leads to a non-terminating chain of integrability conditions. These conditions are highly restrictive, and are likely to hold only in models with special symmetries, such as the known Bianchi and G_2 examples. In models with realistic inhomogeneity, the gravito-magnetic field is necessarily non-transverse at second and higher order.

I. INTRODUCTION

Gravitational waves in cosmology are usually described by transverse traceless tensor perturbations of the background Friedmann-Lemaître-Robertson-Walker (FLRW) metric [1], i.e.

$$ds^2 = a^2(\eta) \left[-d\eta^2 + (\gamma_{ij} + 2\mathcal{H}_{ij}) \, dx^i dx^j \right] \,,$$

where $a^2\gamma_{ij}$ is the background spatial metric, $\gamma^{ij}\mathcal{H}_{ij} = 0 = \mathcal{D}^j\mathcal{H}_{ij}$, and $\mathcal{D}$ is the covariant derivative formed from γ_{ij}. An alternative covariant approach due to Hawking [2] is to describe gravitational radiation via the electric and magnetic parts of the Weyl tensor, which describes the locally free part of the gravitational field. The advantages of this approach are mainly its physical transparency, and the readiness with which one can investigate nonlinear extensions of perturbative theory. In particular, using the covariant approach, one can investigate whether the various characteristic properties of gravitational waves in the perturbative regime carry through to the nonperturbative level. This question is of importance for a covariant understanding of second-order effects in cosmology.

In general terms, we expect that many linearized features will not extend to second order, precisely because of the nonlinearity of the gravitational field, which "creates its own background on which it propagates". One consequence is that the separation of inhomogeneous deviations into scalar, vector and tensor modes is not in general consistent beyond the linearized level, and "mode mixing" occurs [3]. Indeed, in the fully nonlinear regime, it is probably not possible to consistently isolate the radiative part of the gravitational field. Here we investigate whether the transverse property can be maintained at second order – and we find that the answer is negative. We show that *in general the gravito-magnetic field in an irrotational dust universe cannot be transverse in the exact nonlinear theory*. This implies in particular that *at second order, the gravito-magnetic field in cosmology is not in general transverse*. Thus there is unavoidable "contamination" of tensor perturbations with vector and scalar contributions via the divergence.

II. COVARIANT APPROACH TO COSMOLOGICAL GRAVITATIONAL RADIATION

The locally free gravitational field, i.e. that part of the spacetime curvature which is not directly determined locally by the energy-momentum tensor, is given by the Weyl tensor C_{abcd}, which covariantly describes gravitational radiation and tidal forces (see e.g. [2,4–6]). If u^a is the cosmic average 4-velocity (with $u^a u_a = -1$), then comoving observers define the gravito-electric and gravito-magnetic field tensors

$$E_{ab} = C_{acbd}u^c u^d = E_{\langle ab \rangle} \,, \quad H_{ab} = \tfrac{1}{2}\varepsilon_{acd}C^{cd}{}_{be}u^e = H_{\langle ab \rangle} \,,$$

1

which exemplify a remarkable covariant electromagnetic analogy in general relativity [2,5,7]. Here $\varepsilon_{abc} = \eta_{abcd}u^d$ is the spatial alternating tensor formed from the spacetime alternating tensor η_{abcd}, and the angled brackets on indices denote the spatially projected, symmetric and tracefree part:

$$S_{\langle ab \rangle} = \left[h_{(a}{}^c h_{b)}{}^d - \tfrac{1}{3} h^{cd} h_{ab} \right] S_{cd} \,,$$

with $h_{ab} = g_{ab} + u_a u_b$ the spatial projector into the comoving rest space, and g_{ab} the spacetime metric.

In linearized theory, the gravito-electromagnetic fields give a covariant description of the gravitational wave background in cosmology [2,8,9], alternative to the non-covariant description in standard perturbation theory [1,10,11]. A necessary condition for these fields to describe radiation is that their spatial curls should be nonzero; when the field is a pure radiation field, additionally their spatial divergences should vanish, in line with analogous properties of electromagnetic radiation [2,7]. The covariant spatial divergence and curl of tensors are defined by [12]

$$(\operatorname{div} S)_a = \mathrm{D}^b S_{ab} \,, \quad \operatorname{curl} S_{ab} = \varepsilon_{cd(a} \mathrm{D}^c S_{b)}{}^d \,,$$

where the spatial part D_a of the covariant derivative ∇_a is given by

$$\mathrm{D}_a A_{b\cdots} = h_a{}^c h_b{}^d \cdots \nabla_c A_{d\cdots} \,.$$

Wave-propagation requires non-vanishing curls [8,9], i.e.,

$$\operatorname{curl} H_{ab} \neq 0 \neq \operatorname{curl} E_{ab} \,, \tag{1}$$

if there is gravitational radiation present in the spacetime, allowing the linking of gravito-electric and gravito-magnetic variations to each other through the Maxwell-like equations for these quantities, given below. (More precisely, one requires $\operatorname{curl}\operatorname{curl} H_{ab} \neq 0 \neq \operatorname{curl}\operatorname{curl} E_{ab}$. In [6], it is shown that a further necessary condition is nonvanishing distortions, i.e. $\mathrm{D}_{\langle a} H_{bc \rangle} \neq 0 \neq \mathrm{D}_{\langle a} E_{bc \rangle}$. We will not need these refinements here.) In addition, if the gravitational radiation is "pure", i.e., if this radiation encompasses the only deviation of the spacetime (at linear level) from FLRW, then the covariant transverse conditions hold:

$$(\operatorname{div} H)_a = 0 = (\operatorname{div} E)_a \,. \tag{2}$$

This is the simple, transparent and consistent description of perturbative gravitational radiation which flows from the covariant approach [2,8,9].

In the nonperturbative theory, we can impose the covariant conditions, but we cannot a priori expect them to encode the same simple physical meaning. If we can show that any one of the conditions does not in general hold in the nonlinear case, then since these conditions are necessary for pure radiation, that is enough to show a breakdown in the linearized description of gravitational radiation at nonperturbative level. Here we focus on the gravito-magnetic transverse condition $(\operatorname{div} H)_a = 0$, and we show that it is in general inconsistent in the exact nonlinear theory. We do not therefore need to investigate the gravito-electric transverse condition $(\operatorname{div} E)_a = 0$, although all the indications are that this condition also breaks down at nonperturbative level.

We consider irrotational dust spacetimes, which provide reasonable models of the late universe, and allow us to focus on the purely gravitational dynamics of matter. The known exact solutions with $(\operatorname{div} H)_a = 0 \neq \operatorname{curl} H_{ab}$ are spatially homogeneous Bianchi type V, as shown in [12], and G_2 solutions with restricted inhomogeneity, found in [13]. In [14], it was argued that in fact $(\operatorname{div} H)_a = 0$ does not impose any integrability conditions. Although the basic equations and results of [14] are correct, there is a flaw in the concluding argument, as we explain below. A simple integrability condition implied by $(\operatorname{div} H)_a = 0$ was assumed to be satisfied, but it is not identically satisfied in view of other valid equations. Rather it imposes non-trivial conditions which have to be checked. Hence, as shown below, the transverse condition cannot be obeyed in general, although it can be at the linear level.

III. INTEGRABILITY OF THE GRAVITO-MAGNETIC TRANSVERSE CONDITION

The correct statement is that in an irrotational dust universe, $(\operatorname{div} H)_a = 0$ implies the covariant integrability condition

$$\mathcal{I}^{(1)} \equiv [\sigma, \operatorname{curl} H] = 0 \,, \tag{3}$$

where $\sigma_{ab} = \mathrm{D}_{\langle a} u_{b \rangle}$ is the shear, and $[A, B]$ denotes either the tensor commutator of spatial symmetric tensors, i.e. $[A, B]_{ab} \equiv A_{ac} B^c{}_b - B_{ac} A^c{}_b$, or its equivalent spatial dual:

$$[A, B]_a \equiv \varepsilon_{abc} A^b{}_d B^{cd} = \tfrac{1}{2}\varepsilon_{abc}[A, B]^{bc}.$$

To derive Eq. (3) and its consequences, we need some of the covariant propagation and constraint equations, which are [12]:

$$\dot{\rho} = -\Theta\rho \,, \tag{4}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}\rho - \sigma_{ab}\sigma^{ab} \,, \tag{5}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^c - E_{ab} \,, \tag{6}$$

$$\dot{E}_{ab} = -\Theta E_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^c + \operatorname{curl} H_{ab} - \tfrac{1}{2}\rho\sigma_{ab} \,, \tag{7}$$

$$\dot{H}_{ab} = -\Theta H_{ab} + 3\sigma_{c\langle a}H_{b\rangle}{}^c - \operatorname{curl} E_{ab} \,, \tag{8}$$

$$(\operatorname{div}\sigma)_a = \tfrac{2}{3}\mathrm{D}_a\Theta \,, \tag{9}$$

$$\operatorname{curl}\sigma_{ab} = H_{ab} \,, \tag{10}$$

$$(\operatorname{div} E)_a = [\sigma, H]_a + \tfrac{1}{3}\mathrm{D}_a\rho \,, \tag{11}$$

$$(\operatorname{div} H)_a = -[\sigma, E]_a \,, \tag{12}$$

where ρ is the energy density, $\Theta = \mathrm{D}^a u_a$ is the expansion rate, and a dot denotes the covariant time derivative $u^a\nabla_a$. (Note that $\dot{S}_{ab} = \dot{S}_{\langle ab\rangle}$ since the 4-acceleration vanishes.)

The general class of irrotational dust spacetimes is characterized by

$$p = 0 \,, \quad \omega_a = 0 = q_a \,, \quad \pi_{ab} = 0 \,,$$

where p is the isotropic pressure, ω_a is the vorticity, q_a is the energy flux, and π_{ab} is the anisotropic stress. These conditions do not lead to any integrability conditions under time evolution, as shown in [12] (see also [15–18]). By contrast, the class of shear-free dust spacetimes, characterized by

$$p = 0 \,, \quad q_a = 0 \,, \quad \sigma_{ab} = 0 = \pi_{ab} \,,$$

is not in general consistent. Time evolution of the shear-free condition leads to the integrability condition

$$\Theta\omega_a = 0 \,,$$

as shown in [19] (see also [20,21]). More severe conditions arise from imposing restrictions on the gravito-electric/magnetic fields.

In linearized theory, $(\operatorname{div} H)_a$ vanishes by virtue of Eq. (12), while $(\operatorname{div} E)_a = \tfrac{1}{3}\mathrm{D}_a\rho$, i.e. density inhomogeneity is a source for the "Coulomb" part of the gravitational field. In order to isolate the purely radiative part in linearized theory, one sets the scalar inhomogeneity $\mathrm{D}_a\rho$ to zero. (Note that this does not imply there is no inhomogeneity in the matter, only that the inhomogeneity is second order.) Furthermore, in order to remove the remaining scalar and vector modes, one sets $\mathrm{D}_a\Theta = 0$, which by the constraint equation (9) requires $(\operatorname{div}\sigma)_a = 0$. It can then be shown that all these conditions corresponding to purely tensor (radiative) modes do not produce any integrability conditions, i.e. the covariant description is consistent at the linear level [8,9].

At nonperturbative level, $(\operatorname{div} H)_a$ is no longer identically zero. The vanishing of $(\operatorname{div} H)_a$ is equivalent by Eq. (12) to the algebraic condition

$$\mathcal{I}^{(0)} \equiv [\sigma, E] = 0 \,, \tag{13}$$

which is of course identically satisfied at the linear level. This covariant commutation property expresses the fact that one can choose an orthonormal tetrad which is a simultaneous eigenframe of σ_{ab} and E_{ab}. Using the shear propagation equation (6) to eliminate E in condition (13), we find that

$$[\sigma, \dot{\sigma}] = 0 \,,$$

which allows one to further specialize the eigenframe to be Fermi-Walker transported [14]. Conversely, if the shear eigenframe is Fermi-propagated, then Eq. (13) necessarily follows, i.e. the gravito-magnetic field is transverse [14].

It is possible to investigate consistency of the transverse condition (13) using the special tetrad [22], but this leads to the same conclusion, with no less effort, so we prefer to perform a fully covariant analysis in which the physical and geometric quantities themselves are to the fore. The fundamental condition (13) must hold under evolution along u^a. The first time derivative gives

$$\dot{\mathcal{I}}_a^{(0)} = -\tfrac{5}{3}\Theta\mathcal{I}_a^{(0)} - \tfrac{1}{2}\sigma_a{}^b\mathcal{I}_b^{(0)} + \mathcal{I}_a^{(1)}\,, \tag{14}$$

where $\mathcal{I}^{(1)}$ is defined in Eq. (3). To derive equation (14), we commute the shear propagation equation (6) with E_{ab}, and the gravito-electric propagation equation (7) with σ_{ab}, using basic algebraic properties of the commutator. It follows from equation (14) that the first time derivative of $\mathcal{I}^{(0)}$ is not automatically zero by virtue of the original condition (13), but yields the integrability condition (3). Furthermore, this integrability condition does not follow as a consequence of the constraint equations.

Now the time derivative of the primary integrability condition (3) must also be satisfied. Evaluating this derivative requires commuting $\operatorname{curl} H_{ab}$ with the shear propagation equation (6), and σ_{ab} with $(\operatorname{curl} H_{ab})\dot{}$. There is no basic propagation equation for $\operatorname{curl} H_{ab}$. However, the propagation of $\operatorname{curl} H_{ab}$ is indirectly determined by the curl of the gravito-magnetic propagation equation (8), together with the identity [12]

$$(\operatorname{curl} S_{ab})\dot{} = \operatorname{curl} \dot{S}_{ab} - \tfrac{1}{3}\Theta\operatorname{curl} S_{ab} - \sigma_e{}^c\varepsilon_{cd(a}\mathrm{D}^e S_{b)}{}^d + 3H_{c\langle a}S_{b\rangle}{}^c\,. \tag{15}$$

Then using Eqs. (8) and (15), we find that

$$\dot{\mathcal{I}}_a^{(1)} = -2\Theta\mathcal{I}_a^{(1)} - \sigma_a{}^b\mathcal{I}_b^{(1)} - \mathcal{I}_a^{(2)}\,, \tag{16}$$

where

$$\begin{aligned}
\mathcal{I}_a^{(2)} \equiv\ & [\sigma, \operatorname{curl}\operatorname{curl} E]_a - 3[\sigma, \operatorname{curl}\langle\sigma H\rangle]_a + [E, \operatorname{curl} H]_a + 3H_a{}^b[\sigma, H]_b \\
& + \sigma_{bc}H^{bc}\mathrm{D}_a\Theta - \left(H_{ac}\sigma^{cb} + \tfrac{1}{2}\sigma_{ac}H^{cb}\right)\mathrm{D}_b\Theta - \sigma_b{}^d\sigma^{bc}\mathrm{D}_c H_{ad} \\
& + \left(\sigma_a{}^c\sigma_{bd} + \tfrac{1}{2}\sigma_{ab}\sigma^c{}_d\right)\mathrm{D}_c H^{bd}\,.
\end{aligned} \tag{17}$$

Here $\langle AB\rangle_{ab} \equiv (AB)_{\langle ab\rangle} = A_{c\langle a}B_{b\rangle}{}^c$, and we used the properties $\varepsilon_{abc}\varepsilon^{def} = 3!h_{[a}{}^d h_b{}^e h_{c]}{}^f$ and $\varepsilon_{abc}\varepsilon^{dec} = 2h_{[a}{}^d h_{b]}{}^e$. We can rewrite this expression for $\mathcal{I}^{(2)}$ by using the identity [12]

$$\begin{aligned}
\operatorname{curl}\operatorname{curl} S_{ab} =\ & -\mathrm{D}^c\mathrm{D}_c S_{ab} + \tfrac{3}{2}D_{\langle a}(\operatorname{div} S)_{b\rangle} + \left(\rho - \tfrac{1}{3}\Theta^2\right)S_{ab} \\
& + 3S_{c\langle a}\left\{E_{b\rangle}{}^c - \tfrac{1}{3}\Theta\sigma_{b\rangle}{}^c\right\} + \sigma_{cd}S^{cd}\sigma_{ab} - S^{cd}\sigma_{ca}\sigma_{bd} + \sigma^{cd}\sigma_{c(a}S_{b)d}\,,
\end{aligned}$$

and the identity [6]

$$\mathrm{D}_c S_{ab} = \mathrm{D}_{\langle c}S_{ab\rangle} - \tfrac{2}{3}\varepsilon_{dc(a}\operatorname{curl} S_{b)}{}^d + \tfrac{3}{5}(\operatorname{div} S)_{\langle a}h_{b\rangle c}\,.$$

However the expression remains intractable.

Clearly satisfying the primary integrability condition (3) on an initial surface does not in itself guarantee that $\mathcal{I}^{(1)}$ remains zero. Instead, the vanishing of $\dot{\mathcal{I}}^{(1)}$ requires, by Eq. (16), an independent secondary integrability condition

$$\mathcal{I}^{(2)} = 0\,, \tag{18}$$

which is of course identically satisfied at the linear level. The form of Eq. (17) shows that the integrability conditions grow more complicated at each stage, involving a growing order of spatial derivative as well as increasing algebraic complexity. The constraint equations do not lead to any simplification of this integrability condition.

It is clear that further time evolution will produce a third, and then a fourth, fifth, ... integrability condition. Each condition is more complicated than its predecessor, and is not identically satisfied in general by virtue of earlier conditions or the constraint equations. Furthermore, eliminating terms higher than second order does not change this fundamental feature, although it does simplify the integrability conditions somewhat. At second covariant order, the primary integrability condition (3) is unchanged in form, but the secondary integrability condition (18) becomes

$$\mathcal{I}^{(2)} = [\sigma, \operatorname{curl}\operatorname{curl} E] + [E, \operatorname{curl} H] = 0\,. \tag{19}$$

The next integrability condition will involve third order spatial derivatives, and so on.

IV. CONCLUSION

We have shown that the gravito-magnetic transverse condition $(\operatorname{div} H)_a = 0$, taken over from linearized theory into the nonlinear theory, leads to a chain of derived integrability conditions, the first two of which are Eqs. (3) and

(18). This result corrects the assertion in [14] that condition (3) (and thus its derivatives) is identically satisfied. The logical flaw in the argument presented in [14] is essentially the implicit assumption that the propagation equations (6)–(8) are identically satisfied. We have corrected that error here, by imposing the propagation equations as effective conditions on the evolution of $(\operatorname{div} H)_a = 0$, equivalently $[\sigma, E] = 0$.

The consequence of this is that *in general, the gravito-magnetic field cannot be transverse at second (and higher) order, so that it cannot be purely radiative beyond linear order.* In the fully nonlinear case, when one cannot really isolate gravitational radiation, our result implies that the transverse condition is in general inconsistent, i.e. *generic inhomogeneity within the class of irrotational dust spacetimes is inconsistent with a transverse gravito-magnetic field.* These results are independent of gauge choices or of coordinates, since they are fully covariant.

There are however special cases when the integrability conditions are satisfied, the most important of which is the case of almost-FLRW universes. The condition (3) and all its derivatives are satisfied to first order in this case, which simply reflects the fact that the covariant formulation of the transverse condition for perturbative gravitational radiation is consistent at the linear level.

The integrability conditions are also satisfied if $H_{ab} = 0$, which defines the "silent" models. However, in that case, there is another chain of integrability conditions flowing from the vanishing of H_{ab} at all derivative levels. As shown in [23,24], these integrability conditions are even more severe than in the $(\operatorname{div} H)_a = 0$ case. The $H_{ab} = 0$ models are thus in general inconsistent (see also [25]), and unlikely to extend beyond the known special cases where the integrability conditions collapse to identities. These special cases are some Bianchi and Szekeres solutions (see [23,24]). Such a chain of derived integrability conditions of growing complexity exists also in the "anti-Newtonian" models $E_{ab} = 0$ (see [27]), where the only known (irrotational dust) solution is the degenerate FLRW solution. It is clearly very restrictive to impose conditions on the gravito-magnetic and -electric fields of irrotational dust spacetimes.

In the $(\operatorname{div} H)_a = 0 \neq H_{ab}$ case, among the known special exact solutions are Bianchi type V [12] and diagonal (in comoving coordinates) G_2 solutions [13]. In these solutions, the integrability conditions are identically satisfied by virtue of the special structure of $\operatorname{curl} H_{ab}$. If we choose a Fermi-propagated shear eigenframe (as described above), then in these solutions, $\operatorname{curl} H_{ab}$ remains diagonal at all times, so that the primary integrability condition (3) is identically true, as are all its derivatives. For example, for G_2 solutions of the form

$$ds^2 = -dt^2 + A^2(t,x)dx^2 + B^2(t,x)dy^2 + C^2(t,x)dz^2\,,$$

in coordinates that are comoving with the fluid 4-velocity u^a, it follows that

$$H_{ab} = -2C\left(\frac{B_{,x}}{A}\right)_{,t}\delta_{(a}{}^y\delta_{b)}{}^z\,,$$

which leads to a diagonal $\operatorname{curl} H_{ab}$. The Bianchi type V solution has $\operatorname{curl} H_{ab} \propto \sigma_{ab}$ in the shear eigenframe. In the Bianchi case, it is possible to identify further examples [26]. The type V case is distinguished as the only $(\operatorname{div} H)_a = 0 \neq H_{ab}$ solution in Bianchi class B, and it has $(\operatorname{div} E)_a \neq 0$, so that it is not purely radiative in the sense of condition (2). (Note that the diagonal G_2 solutions also have $(\operatorname{div} E)_a \neq 0$.)

Indeed it is clear from our analysis that a similar chain of integrability conditions will flow from the gravito-electric transverse condition in Eq. (2), which is equivalent to

$$\tfrac{1}{3}D_a\rho + [\sigma, H]_a = 0\,.$$

The independent chain of conditions arising from repeated time differentiation of this condition reinforces the unlikelihood of finding inhomogeneous transverse radiative solutions beyond the linear level.

The Bianchi class A solutions all have vanishing divergence for both E_{ab} and H_{ab}, and so provide spatially homogeneous pure radiative examples, except for types I and LRS VII$_0$, which are in the degenerate silent case $H_{ab} = 0$. Thus *in Bianchi class A spacetimes it is possible to find exact solutions satisfying both the radiative transverse conditions (2) and the propagating radiative conditions (1).* We give here two such examples, using the formalism and notation of [30]. In a shear eigenframe $\{\mathbf{e}_\alpha\}$ (where $\alpha = 1, 2, 3$), a tracefree diagonal tensor $S_{\alpha\beta}$ has curl

$$\operatorname{curl} S_{\alpha\beta} = \varepsilon_{\gamma\delta(\alpha}\partial^\delta S_{\beta)}{}^\gamma + \tfrac{1}{2}S_{\alpha\beta}n^\delta{}_\delta - 3n^\gamma{}_{(\alpha}S_{\beta)\gamma} + \delta_{\alpha\beta}S_{\gamma\delta}n^{\gamma\delta}\,, \tag{20}$$

where ∂_α is the directional derivative along $\mathbf{e}_\alpha$, and the tetrad commutation quantities may be diagonalized: $n_{\alpha\beta} = \operatorname{diag}(n_1, n_2, n_3)$. Using the irreducible components $S_+ = -\tfrac{3}{2}S_{11}$ and $S_- = \tfrac{1}{2}\sqrt{3}(S_{22} - S_{33})$, the Bianchi type II solutions (characterized by $n_1 > 0$ and $n_2 = 0 = n_3$) have nonvanishing components of $\operatorname{curl} S_{\alpha\beta}$

$$\operatorname{curl} S_+ = -\tfrac{3}{2}n_1 S_+\,, \quad \operatorname{curl} S_- = \tfrac{1}{2}n_1 S_-\,,$$

and nonvanishing components of $\operatorname{curl}\operatorname{curl} S_{\alpha\beta}$

$$\operatorname{curl}\operatorname{curl} S_+ = \tfrac{9}{4}(n_1)^2 S_+ \,, \quad \operatorname{curl}\operatorname{curl} S_- = \tfrac{1}{4}(n_1)^2 S_- \,.$$

The type VI_0 solutions with $n^\alpha{}_\alpha = 0$ (and $n_2 = -n_1 < 0$, $n_3 = 0$), have nonvanishing components of $\operatorname{curl} S_{\alpha\beta}$

$$\operatorname{curl} S_+ = -\tfrac{1}{2}\sqrt{3}\,n_1(\sqrt{3}S_+ - S_-)\,, \quad \operatorname{curl} S_- = \tfrac{1}{2}\sqrt{3}\,n_1(S_+ + \sqrt{3}S_-)\,,$$

and nonvanishing components of $\operatorname{curl}\operatorname{curl} S_{\alpha\beta}$

$$\operatorname{curl}\operatorname{curl} S_\pm = 3(n_1)^2 S_\pm \,.$$

The tensor $S_{\alpha\beta}$ in both examples may then be chosen to be either the gravito-electric tensor $E_{\alpha\beta}$ or gravito-magnetic tensor $H_{\alpha\beta}$.

The importance of these examples is twofold: (a) they show that exact purely radiative solutions can exist in the minimal covariant sense, and (b) they show that the Petrov type N characterization of a radiative field is more restrictive, because there are no Petrov type N dust metrics in the full nonlinear theory [28,29]. However, because these examples are spatially homogeneous models, information is not being conveyed from place to place by the waves; these are "standing" waves rather than traveling (propagating) waves. By contrast, in the linearized case, arbitrary information can be conveyed by the gravitational waves [9].

It is possible that further, inhomogeneous exact solutions can be found, but they would also be very special cases, in order to satisfy the chain of integrability conditions. Realistic inhomogeneous models cannot be expected to satisfy these conditions.

[1] E.M. Lifshitz, J. Phys. (Moscow) **10**, 116 (1946).
[2] S.W. Hawking, Astrophys. J. **145**, 544 (1966).
[3] S. Matarrese, S. Mollerach, and M. Bruni, Phys. Rev. D **58**, 043504 (1998).
[4] P. Szekeres, J. Math. Phys. **6**, 1387 (1965).
[5] G.F.R Ellis, in *General Relativity and Cosmology*, edited by R.K. Sachs (Academic, New York, 1971).
[6] R. Maartens, G.F.R. Ellis, and S.T.C. Siklos, Class. Quantum Grav. **14**, 1927 (1997).
[7] R. Maartens and B.A. Bassett, Class. Quantum Grav. **15**, 705 (1998).
[8] P.K.S. Dunsby, B.A. Bassett, and G.F.R. Ellis, Class. Quantum Grav. **14**, 1215 (1997).
[9] P.A. Hogan and G.F.R. Ellis, Class. Quantum Grav. **14**, A171 (1997).
[10] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[11] J. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[12] R. Maartens, Phys. Rev. D **55**, 463 (1997).
[13] J.M.M. Senovilla and R. Vera, Class. Quantum Grav. **14**, 3481 (1997).
[14] R. Maartens, W.M. Lesame, and G.F.R. Ellis, Phys. Rev. D **55**, 5219 (1997).
[15] T. Velden, Diplomarbeit, University of Bielefeld, 1997.
[16] M.A.H. MacCallum, gr-qc/9806003, 1998.
[17] H. van Elst and G.F.R. Ellis, Class. Quantum Grav. **15**, 3545 (1998).
[18] R. Maartens, Phys. Rev. D **58**, 124006 (1998).
[19] G.F.R. Ellis, J. Math. Phys. **8**, 1171 (1967).
[20] P. Szekeres, notes, 1996 (unpublished).
[21] J.M.M. Senovilla, C.F. Sopuerta, and P. Szekeres, Gen. Relativ. Gravit. **30**, 389 (1998).
[22] C.F. Sopuerta, notes, 1998 (unpublished).
[23] H. van Elst, C. Uggla, W.M. Lesame, G.F.R. Ellis, and R. Maartens, Class. Quantum Grav. **14**, 1151 (1997).
[24] C.F. Sopuerta, Phys. Rev. D **55**, 5936 (1997).
[25] L. Kofman and D. Pogosyan, Ap. J. **442**, 30 (1995).
[26] W.M. Lesame, Ph.D. thesis, University of Cape Town, 1998.
[27] R. Maartens, W.M. Lesame, and G.F.R. Ellis, Class. Quantum Grav. **15**, 1005 (1998).
[28] W. Kundt and M. Trümper, Akad. Wiss. Mainz **12**, 1 (1962).
[29] P. Szekeres, J. Math. Phys. **7**, 751 (1966).
[30] G.F.R. Ellis and M.A.H. MacCallum, Commun. Math. Phys. **12**, 108 (1969); H. van Elst, Ph.D. thesis, University of London, 1996.